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Power Law Decay of Correlations in a Billiard Problem

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A billiard problem with boundary arcs that meet tangentially is studied both analytically and numerically. It is shown that the presence of tangential vertices leads to velocity correlations which decay like $1/n$ where n is the number of collisions. This result contrasts with related billiard and Lorentz models where velocity correlations decay exponentially.

KEY WORDS: Billiard model; long time tail; velocity correlation function.

1. INTRODUCTION

The present paper reports on a numerical and analytic study of a billiard problem. The model treated here consists of a curved triangular region in which a single point particle moves according to the laws of classical mechanics. The region, shown in Fig. 1, is formed by bringing together three circles, each with unit radius, so that they touch but do not overlap. Within this region the point particle moves with constant velocity and unit speed between elastic collisions with the boundary.

The quantity studied in this work is the velocity correlation, $\phi(n)$, as a function of the number of collisions, n. $\phi(n)$ is defined by

$$\phi(n) = \langle \hat{v}_n \cdot \hat{v}_0 \rangle \tag{1}$$

where $\hat{v}_0$ is the initial velocity and $\hat{v}_n$ is the velocity after the nth collision.

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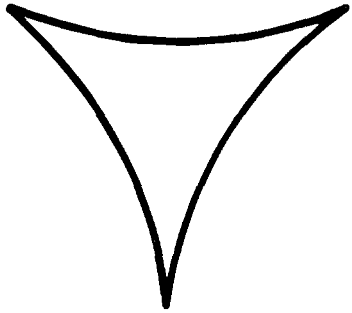

Fig. 1. The region in which the particle moves. The region is formed by bringing three circles of equal radius into contact.

The bracket indicates an average over all initial directions, $\hat{v}_0$ and all initial positions within the region.

The main conclusion of this paper is that, for large n, $|\phi(n)|$ decays like $1/n$ and that for $n > 5$ it is well approximated by

$$\phi(n) \sim \frac{(-1)^n A}{n} \tag{2}$$

where A is about $\frac{1}{2}$.

This power law decay of correlations is due to the fact that the circular arcs forming the boundary of the region meet tangentially. It is thus possible for the moving particle to become trapped for a large number of collisions in the region near one of these tangential vertices. During such a close approach to a vertex, the particle is bouncing between nearly parallel walls and so its velocity is strongly correlated from one collision to the next. The $1/n$ behavior of $\phi(n)$ then arises because the phase space available for remaining near a vertex for n collisions diminishes like $1/n$.

The billiard problem studied here is the limiting case of a periodic Lorentz gas considered by Bunimovich and Sinai.[1] In this Lorentz gas, a single point particle moves in a two-dimensional periodic array of nonoverlapping disks. So long as there are no infinitely long free paths in this array, Bunimovich and Sinai[1] were able to prove that the asymptotic behavior of $\phi(n)$ is dominated by an exponential of the form

$$|\phi(n)| \leqslant e^{-n^{\gamma}} \tag{3}$$

where γ is a constant, $0 < \gamma \leqslant 1$. The present model is obtained from a periodic Lorentz gas on a triangular lattice by letting the diameter of the scatterers equal the lattice spacing.

The present model is also the limiting case of a billiard model in which the region where the particle moves is formed between three overlapping disks. A similar model was studied numerically by Casati *et al.*[2] They considered a billiard in a region formed between four overlapping disks. They mention that the bound given in Eq. (2) should apply to this model and then show, numerically, that a certain correlation function [not $\phi(n)$] indeed decays approximately like $\exp(-1.4n^{0.42})$.

Finally, we remark that asymptotic power law decays in velocity correlations as a function of time ("long-time-tails") are found in fluids,[3] disordered Lorentz models,[4] and disordered random walks.[5] In these cases and the model studied here, the power law decay of correlations is due to scale invariance. While the long-time tails found in disordered systems arise from the self-similarity of large length scales, the "many-collision tail" found here is due to the self-similarity of small length scales.

In Section 2 the class of trajectories which closely approach a vertex is analyzed and shown to yield an asymptotic $1/n$ contribution to $|\phi(n)|$. In section 3 the results of a computer simulation of the model are presented. These results support the conjecture that $|\phi(n)|$ itself decays like $1/n$ for large n. The paper ends with a discussion.

2. ANALYSIS OF THE DYNAMICS NEAR A VERTEX

The motion of the particle in the region shown in Fig. 2 is characterized by a sequence of collisions. Each collision can be described by two angles and a discrete label. Figure 2 shows two collisions and the four

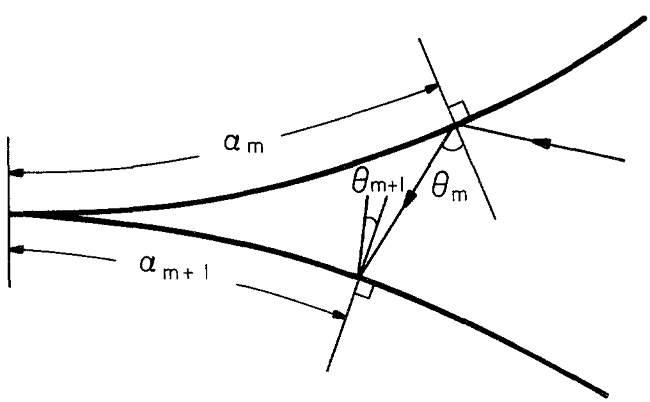

Fig. 2. The mth and $(m+1)$th collisions and the parameters α_m, α_{m+1}, θ_m, and θ_{m+1} used to describe them.

angles used to describe them. α_m specifies the arc length from the mth collision to the nearest vertex. θ_m measures the angle that the velocity makes with the normal at the point of collision. θ_m takes negative values if the particle is approaching the vertex and positive values if it is receding. α_m and θ_m lie in the ranges

$$0 \leqslant \alpha_m \leqslant \pi/6 \tag{4a}$$

and

$$-\pi/2 \leqslant \theta_m \leqslant \pi/2 \tag{4b}$$

The discrete label, σ_m, takes on six values and describes which of six possible boundary faces the collision occurs on. Each face is an arc of the boundary that starts at a vertex and has a length $\pi/6$. In the foregoing analysis we will be concerned with sequences of collisions during which the particle bounces between two faces that adjoin the same vertex. It is convenient then to define an indicator function $\chi(\sigma,\sigma')$ which takes the value 1 if σ and σ' are on adjoining faces and the value 0 otherwise.

Elementary geometric considerations yield the following relation between the mth and $(m+1)$th collisions when $\chi(\sigma_m,\sigma_{m+1})=1$:

$$\sin\alpha_{m+1}-\sin\alpha_m=\left[2-\cos\alpha_{m+1}-\cos\alpha_m\right]\tan(\theta_m+\alpha_m) \tag{5a}$$

$$\theta_{m+1}-\theta_m=\alpha_{m+1}+\alpha_m \tag{5b}$$

Similar considerations show that

$$\theta_m \leqslant \pi/2-4\alpha_m \tag{6}$$

is sufficient to ensure that $\chi(\sigma_m,\sigma_{m+1})=1$.

We shall now calculate the asymptotic contribution to $\phi(n)$ of the class of collision sequences which remain near a single vertex. Call this contribution $\phi^*(n)$. It is defined by

$$\begin{aligned}\phi^*(n) &= \left\langle \hat{v}_n\cdot\hat{v}_0\left[\sum_\sigma \chi(\sigma_0,\sigma)\prod_{m=0}^{n-1}\chi(\sigma_m,\sigma_{m+1})\right]\right\rangle \\ &= (-1)^n\sum_\sigma\int_{-\pi/2}^{\pi/2} f(\theta_0)\,d\theta_0\int_0^{\pi/6} g(\alpha_0)\,d\alpha_0\,\chi(\sigma_0,\sigma) \\ &\quad\times\prod_{m=0}^{n-1}\chi(\sigma_m,\sigma_{m+1})\cos\left[\theta_n-(-1)^n\theta_0\right]\end{aligned} \tag{7}$$

$f(\theta_0)\,d\theta_0$ and $g(\alpha_0)\,d\alpha_0$ are the natural measures for θ_0 and α_0, respectively,

$$f(\theta_0)=\tfrac{1}{2}\cos(\theta_0) \tag{8a}$$

and

$$g(\alpha_0)=1/\pi \tag{8b}$$

The quantity in the square brackets in Eq. (7) restricts the average to those trajectories where the particle bounces between the two faces connected to a single vertex.

Time reversal symmetry permits us to restrict the averaging to those trajectories for which $|\theta_n| \leqslant |\theta_0|$ so long as the average is then multiplied by 2. Since θ_m is an increasing sequence near a single vertex, the upper limit of the θ_0 integration can now be set to 0. The equivalence of the six faces allows us to replace the sum over σ by fixing σ_0 and multiplying by a factor of 6. Thus an equivalent expression for $\phi^*(n)$ is

$$\phi^*(n) = \frac{6(-1)^n}{\pi} \int_{-\pi/2}^{0} \cos\theta_0 \, d\theta_0 \int_0^{\pi/6} d\alpha_0 \times \prod_{m=0}^{n-1} \chi(\sigma_m, \sigma_{m+1}) h(-\theta_0 - \theta_n) \cos\left[\theta_n - (-1)^n \theta_0\right] \tag{9}$$

where $h(x)$ is the unit step function.

We will now show that, for large n, the dominant contribution to $\phi^*(n)$ arises from trajectories where α_m is of order $1/n$ and $\pi/2 - |\theta_m| \gg \alpha_m$ for $m = 0, 1, \ldots, n$. To justify this claim and to find the asymptotic behavior of $\phi^*(n)$, let us analyze the behavior of trajectories in this regime. For α_0 small and θ_m not too near $\pi/2$, the difference equations which describe the motion, Eqs. (5a) and (5b), reduce to differential equations by making the substitutions

$$t \equiv m\alpha_0 \tag{10a}$$

$$y(t) \equiv \frac{\alpha_{t/\alpha_0}}{\alpha_0} \tag{10b}$$

and

$$\theta(t) \equiv \theta_{t/\alpha_0} \tag{10c}$$

In the limit $\alpha_0 \to 0$ holding θ_m fixed we obtain

$$\frac{dy}{dt} = y^2 \tan\theta \tag{11a}$$

and

$$\frac{d\theta}{dt} = 2y \tag{11b}$$

In the regime where α_0 and θ_0 are both small these differential equations yield good approximations to the exact dynamics so long as

$$\frac{\pi}{2} - |\theta_m| \gg \alpha_m \tag{12}$$

otherwise the singularity in the tangent on the right-hand side of Eq. (5a) will be governed by both α_m and θ_m.

Combining Eqs. (11a) and (11b) and integrating yields the following relation between $y(t)$ and $\theta(t)$:

$$y(t)=\left[\frac{\cos\theta_0}{\cos\theta(t)}\right]^{1/2} \tag{13}$$

Combining this relation with Eq. (11b) and integrating yields the following expression for $\theta(t)$:

$$\int_{\theta_0}^{\theta(t)}(\cos x)^{1/2}\,dx=2t(\cos\theta_0)^{1/2} \tag{14}$$

To simplify the notation, define the elliptic integral,

$$I(\theta)=\int_0^{\theta}(\cos x)^{1/2}\,dx \tag{15}$$

The formal solution to Eq. (14) is then

$$\theta(t)=I^{-1}\left[2tI'(\theta_0)+I(\theta_0)\right] \tag{16}$$

For $\theta_0<0$ the qualitative behavior of the solution is that $\theta(t)$ is monotonically increasing while $y(t)$ diminishes to a minimum and then increases.

Let

$$\tau=\alpha_0 n \tag{17}$$

The restriction that $\theta_n \leqslant -\theta_0$ sets an upper bound on τ. Solving Eq. (16) for τ with $\theta(\tau)=-\theta_0$ and noting that I' is an even function whereas I is an odd function we obtain the upper bound $T(\theta_0)$,

$$\tau\leqslant -I(\theta_0)/I'(\theta_0)\equiv T(\theta_0) \tag{18}$$

For fixed n, this inequality sets an upper limit on the α_0 integration in Eq. (9),

$$\alpha_0\leqslant\frac{-I(\theta_0)}{nI'(\theta_0)} \tag{19}$$

and explains why the phase space available for very long trajectories near a vertex diminishes like $1/n$.

Using the same kind of reasoning, an n dependent lower bound for θ_0 can be found which ensures that $\pi/2-|\theta_m|\gg\alpha_m$. Let $a(n)-\pi/2$ be a lower limit of integration for an estimate of the θ_0 integral in Eq. (9). We require that $a(n)\gg\alpha_0$, so from Eq. (19) we obtain

$$a(n)^{3/2}\gg I(\pi/2)/n \tag{20}$$

by expanding the cosine near $\pi/2$. On the other hand, we would like to estimate $\phi^*(n)$ in a way that becomes exact when $n\to\infty$. We can satisfy Eq. (20) and yet make a vanishingly small error in the θ_0 integration by choosing $a(n)$ so the $a(n)\to 0$ but $n^{2/3}a(n)\to\infty$ as $n\to\infty$.

Employing the substitution of Eq. (17) and taking the limit $n \to \infty$ we obtain

$$\lim_{n\to\infty}(-1)^n n\phi^*(n) = \lim_{n\to\infty}\frac{6}{\pi}\int_{-\pi/2+a(n)}^{0}\cos\theta_0\,d\theta_0\int_0^{T(\theta_0)}d\tau\cos\left[\theta(\tau)-(-1)^n\theta_0\right]+R \tag{21}$$

from Eq. (9). Note that the condition of Eq. (6) is satisfied so that the χ factors can be omitted from the averaging. The remainder term takes the form

$$R = \lim_{n\to\infty}\frac{6n}{\pi}\int_{-\pi/2}^{-\pi/2+a(n)}\cos\theta_0\,d\theta_0\int_0^{\pi/6}d\alpha_0 \times\prod_{m=0}^{n-1}\chi(\sigma_m,\sigma_{m+1})h(-\theta_0-\theta_n)\cos\left[\theta_n-(-1)^n\theta_0\right] \tag{22}$$

and is bounded by

$$R \leqslant \lim_{n\to\infty} na(n)^2 \tag{23}$$

Thus $a(n)$ can be chosen so that R vanishes and Eq. (20) is satisfied.

In the Appendix it is shown that the terms odd in $\theta(\tau)$ in Eq. (21) vanish on integration so that we obtain

$$\lim_{n\to\infty}(-1)^n n\phi^*(n) = \frac{6}{\pi}\int_{-\pi/2}^{0}\cos^2\theta_0\,d\theta_0\int_0^{T(\theta_0)}d\tau\cos\theta(\tau) = D \tag{24}$$

D is evaluated in the appendix and has the value $D = 0.7295$.

3. COMPUTER RESULTS

In this section, the results of a computer simulation of the billiard model are presented. In the computer experiment six trajectories were generated, each of 100,000 collisions. Using the presumed ergodicity of the model, $\phi(n)$ was computed from each trajectory by averaging $\hat{v}_{n+k}\cdot\hat{v}_k$ over k. Figure 3 shows $(-1)^n n\phi(n)$ versus n for the range $1 \leqslant n \leqslant 30$. Each point represents an average over the six runs. The error bars are the standard deviation. The spread in the data for odd n was considerably less than for even n as illustrated in Fig. 3 by neighboring pairs of error bars. The average of $(-1)^n n\phi(n)$ in the range $5 \leqslant n \leqslant 30$ yields the value 0.54.

The computer results are consistent with the hypothesis that $\phi(n)$ decays like $(-1)^n/n$ for large n, although other functional forms cannot be ruled out. The oscillation in sign in $\phi(n)$ suggests that the main contribution

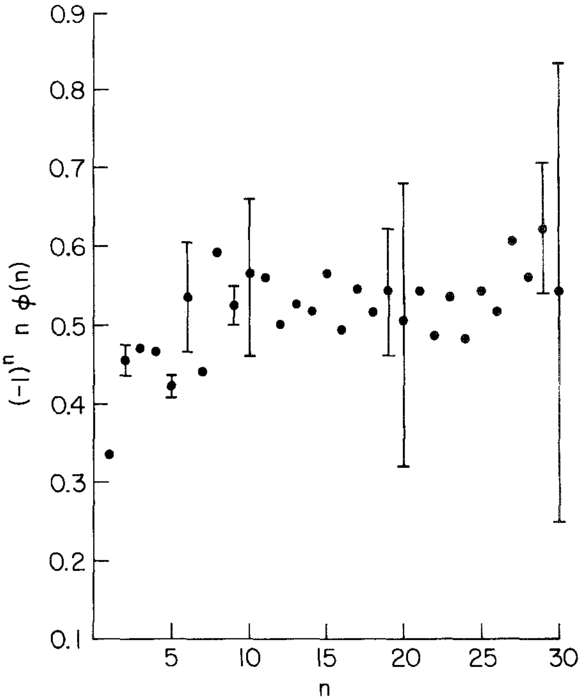

Fig. 3. The velocity correlation function times $(-1)^n n$ vs. n.

to $\phi(n)$ arises from collisions sequences near a vertex, in agreement with the analysis of the previous section. On the other hand, the observed coefficient of 0.54 for the $(-1)^n/n$ decay of $\phi(n)$ is significantly smaller than the coefficient, $D = 0.73$ found from the asymptotic analysis of those sequences of collisions that remain close to a single vertex.

4. DISCUSSION

In the second section we analyzed the class of collision sequences of length n which remain close to one of the vertices. We found the contribution of these collision sequences to the velocity correlation function, $\phi(n)$. This contribution, called $\phi^*(n)$, behaves asymptotically like

$$\lim_{n\to\infty} (-1)^n \phi^*(n) n = D$$

with $D = 0.7295$. Thus $\phi(n)$ itself decays at least as slowly as $1/n$ unless the contribution from $\phi^*(n)$ is fortuitously cancelled by some other class of collision sequences.

The results of the computer experiment support the hypothesis that $\phi(n)$ indeed decays like $(-1)^n/n$. Thus, I conjecture that

$$\lim_{n\to\infty}(-1)^n n\phi(n) = A$$

and that A is about one half.

While $\phi^*(n)$ and $\phi(n)$ behave in qualitatively the same way for large n, the coefficient of A extrapolated from the computer experiment does not agree with the coefficient D determined in the second section. One explanation for this is that the computer experiment has not yet probed the asymptotic regime. Another explanation, which I prefer, is that there is an additional class of collision sequences which contribute an asymptotic $1/n$ term to $\phi(n)$. I believe that these additional trajectories include repeated close encounters to the same vertex. Each encounter is separated by a collision on the wall opposite the vertex.

The qualitative features of the analysis of the second section should hold more generally for any billiard model with tangentially meeting vertices. The only requirement is that at least one of the boundary arcs at each tangential vertex must have a nonzero curvature. The differential equations which describe the dynamics very near a particular vertex are the same as Eqs. (11a) and (11b) except that the right-hand side of each equation is multiplied by $(\kappa_1+\kappa_2)/2$. κ_1 and κ_2 are the curvatures of the two boundary arcs at the vertex. Thus, the long time behavior of $\phi^*(n)$ is generally of the form $D'(-1)^n/n$. D' will depend on the number of tangential vertices, the total length of the boundary and the values of the curvature of the boundary arcs at each vertex.

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APPENDIX

We wish to evaluate the integral

$$D = \frac{6}{\pi}\int_{-\pi/2}^{0}\cos\theta_0\,d\theta_0\int_0^{T(\theta_0)}d\tau\cos\left[\theta(\tau)-(-1)^n\theta_0\right] \tag{A.1}$$

and to show that it does not depend on n. The n-dependent factor in the integrand is

$$\cos\left[\theta(\tau)-(-1)^n\theta_0\right] = \cos\theta(\tau)\cos\theta_0 + (-1)^n\sin\theta(\tau)\sin\theta_0 \tag{A.2}$$

Thus D will be independent of n if, for all θ_0,

$$\int_0^{T(\theta_0)} d\tau \sin\theta(\tau) = 0 \tag{A.3}$$

Let $\tau' = \tau - T(\theta_0)/2$; from Eqs. (16) and (18) we have

$$\theta(\tau) = I^{-1}\left[2\tau' I'(\theta_0)\right] \tag{A.4}$$

Since $I^{-1}(x)$ and $\sin(x)$ are both odd in x,

$$\int_{-T(\theta_0)/2}^{T(\theta_0)/2} d\tau' \sin I^{-1}\left[2\tau' I'(\theta_0)\right] = 0 \tag{A.5}$$

and Eq. (A.3) is proved. Thus D reduces to

$$D = \frac{6}{\pi}\int_{-\pi/2}^{0} d\theta_0 (\cos\theta_0)^2 \int_{-T(\theta_0)/2}^{T(\theta_0)/2} d\tau' \cos I^{-1}\left[2\tau'(\cos\theta_0)^{1/2}\right] \tag{A.6}$$

Making the substitution

$$x = I^{-1}\left[2\tau'(\cos\theta_0)^{1/2}\right]$$

yields

$$D = \frac{3}{\pi}\int_{-\pi/2}^{0} d\theta_0 (\cos\theta_0)^{3/2} \int_{-\theta_0}^{\theta_0} dx (\cos x)^{3/2} \tag{A.7}$$

Using the result

$$\int_0^{\pi/2} (\cos x)^{3/2} dx = \frac{\pi}{5(2^{3/2}) B(7/4, 7/4)} \tag{A.8}$$

we obtain

$$D = 0.7295 \tag{A.9}$$

NOTE ADDED IN PROOF

The argument of the final cosines in Eqs. 7 and 9 should be:

$$\left[(\theta_n - \alpha_n) - (-1)^n(\theta_0 - \alpha_0)\right]$$

These omitted terms should be included in the remainder, R of Eq. 21. It is straightforward to verify that they vanish in the limit $n \to \infty$. Thus the asymptotic result, Eq. 24, is unaffected.

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